Application of Rayleigh-Ritz Method to Dielectric Steps in Waveguides*

R. E. COLLIN† AND R. M. VAILLANCOURT†

Summary—**The Rayleigh-Ritz method is applied to obtain approximations to the first N eigenfunctions and corresponding eigenvalues in an inhomogeneously filled rectangular waveguide. These approximate eigenfunctions are then used to obtain a solution for the reflection and transmission coefficients at the junction of an empty and partially filled waveguide. Theoretical and experimental results are given for a dielectric slab which extends completely across the broad dimension of the guide, but only partially across the narrow dimension. The experimental values are within the experimental error of the computed values obtained by considering the dominant mode and only two evanescent modes.**

INTRODUCTION

WAVE propagation in a waveguide inhomogeneously filled with a dielectric has been studied by many authors.[1] As a general rule, the modes are more complex and transcendental equations have to be solved, in order to find the propagation constants of the various modes. This has led several authors to consider the application of variational methods for obtaining approximations to the eigenvalues.[2–4] By means of the Rayleigh-Ritz method (hereafter called the R-R method), one may obtain approximations for the first N eigenvalues, and also for the first N eigenfunctions.[5,6] In this paper, the discontinuity between an empty and an inhomogeneously filled rectangular guide will be studied using the R-R method. The following procedure is used:

1) The first N eigenfunctions in the inhomogeneously filled guide are approximated by the Rayleigh-Ritz method.
2) Equations expressing the continuity of the tangential field components at the junction are then easily written down and solved for the reflection and transmission coefficients.

In a rectangular guide inhomogeneously filled with a dielectric slab, as in Fig. 1, the two sets of fundamental modes are the longitudinal section electric and magnetic modes (LSE and LSM modes), having the electric and magnetic vector, respectively, contained entirely within a longitudinal section. In an empty guide, the TE and TM modes may be derived from a magnetic and an electric Hertzian potential having only a longitudinal component respectively.[7] By analogy with this problem, it is readily seen that the LSE and LSM modes may be derived from a magnetic and an electric Hertzian potential, respectively and having a single component directed normal to the dielectric-empty guide interface. These two sets of modes form a complete set in which any arbitrary field distribution may be expanded.[8]

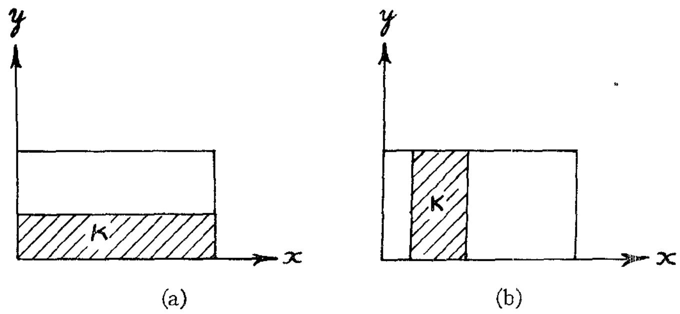

Fig. 1—Inhomogeneously filled rectangular waveguide.

For a general inhomogeneously filled cylindrical guide, the division into LSE and LSM modes is not possible.

When an H_{10} mode is incident at the junction of an empty rectangular guide and a guide partially filled, as in Fig. 1(a), both LSE and LSM modes are excited. In the case of Fig. 1(b) with an H_{10} mode incident, only H_{no} modes are excited. The case of Fig. 1(a) is considerably more complex and, therefore, was chosen as a good example to which the R-R method could be applied.

This same type of dielectric step discontiuity has been treated in a recent paper by Angulo, using the correct expressions for the eigenfunctions and a variational method for evaluation of the equivalent circuit parameters.[9] However, the coupling of the LSE modes by the

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† Canadian Armament Res. and Dev. Establishment, Valcartier, P. Q.

[1] L. G. Chambers, "Propagation in waveguides filled longitudinally with two or more dielectrics," *Brit. J. Appl. Phys.*, vol. 4, pp. 39–45; February, 1953. (This is a review article containing sixteen references.)

[2] L. G. Chambers, "Compilation of the propagation constants of an inhomogeneously filled waveguide," *Brit. J. Appl. Phys.*, vol. 3, pp. 19–21; January, 1952.

[3] L. G. Chambers, "An approximate method for the calculation of propagation constants for inhomogeneously filled waveguides," *Quart. J. Mech. and Appl. Math.*, vol. 7, pt. 3, pp. 299–316; September, 1954.

[4] A. D. Berk, "Variational principles for electromagnetic resonators and waveguides," IRE TRANS., vol. AP-4, pp. 104–111; April, 1956.

[5] R. Courant and D. Hilbert, "Methods of Mathematical Physics," Interscience Publishing Co., New York, N. Y., 1st English ed., p. 175; 1953.

[6] R. Weinstock, "Calculus of Variations," McGraw-Hill Book Co., Inc., New York, N. Y., 1st ed., chs. 7–9; 1952.

[7] J. A. Stratton, "Electromagnetic Theory," McGraw-Hill Book Co., Inc., New York, N. Y., sec. 6.1; 1941.

[8] J. Van Bladel, "Field expandibility in normal modes for a multilayered rectangular or circular waveguide," *J. Franklin Inst.*, vol. 253, pp. 313–321; April, 1952.

[9] C. M. Angulo, "Discontinuities in a rectangular waveguide partially filled with dielectric," IRE TRANS., vol. MTT-5, pp. 68–74; January, 1957.

step when a LSM mode is incident is neglected. Therefore, the results obtained are only approximate and valid for small discontinuities; *i.e.* ($t \ll b$ or $t \approx b$, where t is the slab thickness and b the waveguide height) when the amplitudes of the coupled LSE modes are small.

This same step has been analyzed by R. E. Collin (unpublished work) as well as the simpler case of the H-plane dielectric step.[10]

Use of the R-R method has the great advantage of avoiding evaluation of many complicated expressions, especially for cases where a waveguide cross section is divided into more than two regions by dielectric media of different dielectric constant.

Wave Equation for the Modes

In order to bring the problem being studied into the class that can be handled by the normal Sturm-Liouville theory, the rectangular guide will be considered as filled with a lossless dielectric material with a dielectric constant which is a continuous function of the coordinate y, but independent of x and z. From the theory for the continuous case, one may readily pass to the case where the dielectric constant is a discontinuous function of y; *e.g.*, a slab filling. With a varying dielectric constant, the wave equation satisfied by the Hertzian potentials are modified and therefore will be derived here.

LSM Modes

Maxwell's equations in a source free medium are

$$\nabla \times \vec{H} = j\omega\epsilon_0\kappa\vec{E}, \tag{1a}$$
$$\nabla \times \vec{E} = -j\omega\mu\vec{H}, \tag{1b}$$
$$\nabla \cdot \vec{B} = 0, \tag{1c}$$
$$\nabla \cdot \kappa E = 0, \tag{1d}$$

where κ is the relative dielectric constant and here is considered as a function of y.

By virtue of (1c), one may take

$$\vec{H} = j\omega\epsilon_0\nabla \times \vec{\Pi}_E \tag{2}$$

where $\vec{\Pi}_E$ is an electric Hertzian potential with a y component only. From (1b)

$$\nabla \times \vec{E} = k_0^2\nabla \times \vec{\Pi}_E$$

which integrates to

$$\vec{E} = k_0^2\vec{\Pi}_E + \nabla\phi. \tag{3}$$

From (1a)

$$\nabla \times \nabla \times \vec{\Pi}_E = \nabla\nabla \cdot \vec{\Pi}_E - \nabla^2\vec{\Pi}_E = \kappa k_0^2\vec{\Pi}_E + \kappa\nabla\phi$$
$$= \kappa k_0^2\vec{\Pi}_E + \nabla\kappa\phi - \phi\nabla\kappa. \tag{4}$$

As yet $\nabla\phi$ and $\nabla \vec{\Pi}_E$ are unspecified and one may, therefore, put

$$\nabla\nabla \cdot \vec{\Pi}_E = \nabla\kappa\phi$$

which apart from an irrelevant constant integrates to

$$\nabla \cdot \vec{\Pi}_E = \kappa\phi. \tag{5}$$

From (5), $\phi = \kappa^{-1}\nabla \cdot \vec{\Pi}_E$ and hence the wave equation for $\vec{\Pi}_E$, *i.e.*, (4) becomes

$$\nabla^2\vec{\Pi}_E + \kappa k_0^2\vec{\Pi}_E - \kappa^{-1}\nabla\kappa\nabla \cdot \vec{\Pi}_E = 0 \tag{6}$$

while (3) for the electric field becomes

$$\begin{aligned}\vec{E} &= k_0^2\vec{\Pi}_E + \nabla(\kappa^{-1}\nabla \cdot \vec{\Pi}_E)\\ &= \nabla \cdot \vec{\Pi}_E\nabla\kappa^{-1} + \kappa^{-1}\nabla\nabla \cdot \vec{\Pi}_E + \kappa k_0^2\vec{\Pi}_E\\ &= \kappa k_0^2\vec{\Pi}_E + \kappa^{-1}\nabla\nabla \cdot \vec{\Pi}_E - \kappa^{-2}\nabla \cdot \vec{\Pi}_E\nabla\kappa\\ &= \kappa^{-1}\nabla \times \nabla \times \vec{\Pi}_E,\end{aligned} \tag{7}$$

this latter result following from the wave equation. From (7), one sees at once that (1d) is satisfied, since

$$\nabla\ \kappa\vec{E} = \nabla \cdot (\nabla \times \nabla \times \vec{\Pi}_E) \equiv 0.$$

For a variation with x according to $\sin \pi x/a$ and exponential z dependence, the solutions to (6) are of the form

$$\vec{\Pi}_E = \vec{i}_y \sin\frac{\pi x}{a}\psi_E(y)e^{\pm\gamma z} \tag{8}$$

where ψ_E is a solution of the Sturm-Liouville equation

$$\frac{d^2\psi_E}{dy^2} - \kappa^{-1}\frac{d\kappa}{dy}\frac{d\psi_E}{dy} + \left(\kappa k_0^2 - \frac{\pi^2}{a^2} + \gamma^2\right)\psi_E = 0, \tag{9}$$

or equivalently

$$\frac{d}{dy}\frac{1}{\kappa}\frac{d\psi_E}{dy} + \frac{1}{\kappa}\left(\kappa k_0^2 - \frac{\pi^2}{a^2} + \gamma^2\right)\psi_E = 0.$$

Eq. (9) has an infinite number of solutions ψ_{En} with corresponding eigenvalues γ_n^2. These solutions form an orthogonal set with respect to the weighting function κ^{-1} and may be normalized so that

$$\int_0^b \psi_{En}\psi_{Es}\kappa^{-1}dy = \delta_{ns} \tag{10}$$

where δ_{ns} is the Kronecher delta and is equal to unity, if $n = s$ and zero otherwise. When κ is a continuous function, both ψ_{En} and $d\psi_{En}/dy$ are continuous.

When κ is the discontinuous function,

$$\kappa(y) = \kappa_0 - (\kappa_0 - 1)U(y - t) \tag{11}$$

where $U(y-t)$ is the step function

$$U(y - t) = \begin{cases}0, & y < t,\\ 1, & y \geqq t,\end{cases} \tag{12}$$

[10] R. E. Collin and J. Brown, "The calculation of the equivalent circuit of an axially unsymmetrical waveguide junction," *Proc. IEE*, vol. 103, pt. C, pp. 121–128; March, 1956.

ψ_{En} and $\kappa^{-1}(d\psi_{En}/dy)$ are continuous in order that the tangential field components should be continuous at the interface. At $y=o$, b, $d\psi_{En}/dy$ vanishes. The term

$$\kappa^{-1}\frac{d\kappa}{dy}\frac{d\psi_E}{dy}$$

in the differential equation (9) becomes

$$\kappa^{-1}(1-\kappa_0)\delta(y-t)\frac{d\psi_E}{dy},$$

where $\delta(y-t)$ is the Dirac impulse function. The term $d\psi_E/dy$ is discontinuous at $y=t$ and hence, the second derivative also has an impulse discontinuity at $y=t$. The physical reason for these discontinuities is the polarization charge in the dielectric.

LSE Modes

The longitudinal section electric modes do not have a component of electric field parallel to $\nabla\kappa$ and, therefore,

$$\nabla\cdot\kappa\vec{E} = \kappa\nabla\cdot\vec{E} + \vec{E}\cdot\nabla\kappa = 0$$

gives $\nabla\cdot\vec{E}=0$. For these modes, one may take

$$\vec{E} = -j\omega\mu\nabla\times\vec{\Pi}_M \tag{13}$$

where $\vec{\Pi}_M$ is a magnetic Hertzian potential with only a y component. From (1a),

$$\nabla\times\vec{H} = \kappa k_0^2\nabla\times\vec{\Pi}_M = k_0^2\nabla\times\kappa\vec{\Pi}_M,$$

since

$$\nabla\times\kappa\vec{\Pi}_M = \kappa\nabla\times\vec{\Pi}_M - \vec{\Pi}_M\times\nabla\kappa = \kappa\nabla\times\vec{\Pi}_M.$$

This equation integrates to

$$\vec{H} = \kappa k_0^2\vec{\Pi}_M + \nabla v.$$

From (1b),

$$\nabla\nabla\cdot\vec{\Pi}_M - \nabla^2\vec{\Pi}_M = \kappa k_0^2\vec{\Pi}_M + \nabla v.$$

Let $\nabla\nabla\cdot\vec{\Pi}_M = \nabla v$ and the wave equation for $\vec{\Pi}_M$ becomes

$$\nabla^2\vec{\Pi}_M + \kappa k_0^2\vec{\Pi}_M = 0, \tag{14}$$

while the equation for $\vec{H}$ becomes

$$\vec{H} = \kappa k_0^2\vec{\Pi}_M + \nabla\nabla\cdot\vec{\Pi}_M = \nabla\times\nabla\times\vec{\Pi}_M. \tag{15}$$

With an x variation according to $\cos\pi x/a$ and exponential z variation, the solution to (14) is of the form

$$\vec{\Pi}_M = \vec{i}_y\cos\frac{\pi x}{a}\psi_M(y)e^{\pm\beta z}, \tag{16}$$

where ψ_M is a solution of

$$\frac{d^2\psi_M}{dy^2} + \left(\kappa k_0^2 - \frac{\pi^2}{a^2} + \beta^2\right)\psi_M = 0. \tag{17}$$

Eq. (17) has an infinite number of solutions, ψ_{Mn}, with corresponding eigenvalues, β_n^2, and these solutions can be chosen to form an orthonormal set such that

$$\int_0^b \psi_{Mn}\psi_{Ms}dy = \delta_{ns}. \tag{18}$$

Both ψ_{Mn} and $d\psi_{Mn}/dy$ are continuous, irrespective of whether κ is continuous or not. At $y=0$, b, ψ_{Mn} vanishes.

Minimum Characterization of the Eigenvalues

LSM Modes

If (9) is multiplied by $\kappa^{-1}\psi_E$ and the term involving the second derivative integrated by parts once (this term vanishes, since ψ_E and $\kappa^{-1}(d\psi_E/dy)$ are both continuous and $d\psi_E/dy$ vanishes at $y=0$, b), one gets

$$\gamma^2\int_0^b\kappa^{-1}\psi_E^2dy - \int_0^b\left\{\left(\frac{d\psi_E}{dy}\right)^2 - \left(\kappa k_0^2 - \frac{\pi^2}{a^2}\right)\psi_E^2\right\}\kappa^{-1}dy = 0. \tag{19}$$

Eq. (19) is a variational expression for the propagation constant γ^2. An extremisation of this equation by that class of functions $\phi(y)$, which are continuous with at least a piecewise continuous derivative and orthogonal to the first $K-1$ correct eigenfunctions ψ_{Ei}, with respect to the weight factor κ^{-1}, yields an upper bound on the true eigenvalue γ_K^2. By means of the expansion theorem for a complete set of functions, one may write

$$\phi = \sum_0^\infty a_n\psi_{En}. \tag{20}$$

The orthogonalization conditions give

$$a_n = 0; \qquad n = 0, 1, \cdots K-1. \tag{21}$$

Substituting into (19), gives

$$\begin{aligned}\gamma^2\sum_K^\infty a_n^2 &= \int_0^b\kappa^{-1}\sum_{s=K}^\infty\sum_{n=K}^\infty\left\{\frac{d\psi_{Es}}{dy}\frac{d\psi_{En}}{dy} - \left(\kappa k_0^2 - \frac{\pi^2}{a^2}\right)\psi_{Es}\psi_{En}\right\}a_na_sdy \\ &= \sum_{s=K}^\infty\sum_{n=K}^\infty a_sa_n\left\{\left(\kappa^{-1}\psi_{Es}\frac{d\psi_{En}}{dy}\right)\Big|_0^b - \int_0^b\kappa^{-1}\left[\psi_{Es}\frac{d^2\psi_{En}}{dy^2} + \left(\kappa k_0^2 - \frac{\pi^2}{a^2}\right)\psi_{Es}\psi_{En}\right]dy\right\}.\end{aligned}$$

The integrated term vanishes and, using (9) and (10), the result is

$$\gamma^2\sum_{n=K}^\infty a_n^2 = \sum_{n=K}^\infty a_n^2\gamma_n^2. \tag{22}$$

Assuming that the eigenfunctions ψ_{En} have been ordered so that $\gamma_0^2 < \gamma_1^2 < \gamma_2^2 \cdots < \gamma_K^2$, the result (22) may be written as

$$\gamma^2 = \frac{\sum_{n=K}^{\infty} a_n^2 \gamma_n^2}{\sum_{n=K}^{\infty} a_n^2} = \gamma_K^2 + \frac{\sum_{n=K+1}^{\infty} a_n^2(\gamma_n^2 - \gamma_K^2)}{\sum_{n=K}^{\infty} a_n^2} \geq \gamma_K^2, \quad (23)$$

since $\gamma_n^2 > \gamma_K^2$ for $n > K$. Only when $\phi \equiv \psi_{EK}$, will $\gamma^2 = \gamma_K^2$. In general, the approximate eigenvalue is too large. A suitable series of functions to use for this extremisation are the corresponding eigenfunctions for the empty guide. For the LSM modes these are

$$\phi_{En} = \sqrt{\frac{\epsilon_{on}}{b}} \cos \frac{n\pi}{b} y, \quad n = 0, 1, 2, \cdots, \quad (24)$$

where ϵ_{on} is the Neumann factor,

$$\epsilon_{on} = \begin{cases} 1, & n = 0 \\ 2, & n > 0. \end{cases}$$

LSE Modes

For the LSE modes, the variational expression corresponding to (19) is

$$\beta^2 \int_0^b \psi_M^2 dy - \int_0^b \left\{ \left(\frac{d\psi_M}{dy} \right)^2 - \left(\kappa k_0^2 - \frac{\pi^2}{a^2} \right) \psi_M^2 \right\} dy = 0. \quad (25)$$

As for the previous case, the approximation to the Kth eigenvalue by that class of functions which are continuous and vanish at $y=0$, b, and are orthogonal to the first $K-1$ eigenfunctions ψ_{Mi} is from above. A suitable set of functions for this extremisation are again the corresponding functions for the empty guide, *i.e.*,

$$\phi_{Mn} = \sqrt{\frac{2}{b}} \sin \frac{n\pi}{b} y, \quad n = 1, 2, \cdots. \quad (26)$$

The above variational expressions are also valid when κ is a discontinuous function of y.

The Approximate Eigenfunctions

This section will consider the solution for the first $N+1$ approximate eigenvalues and corresponding approximate eigenfunctions for the case of the LSM modes. In the previous section, it was shown that the extremisation of (19), with respect to functions which were orthogonal to the first $K-1$ true eigenfunctions, gave an upper bound on the Kth eigenvalue. Since one does not know the true eigenfunctions, the class of functions to be used for the Kth extremisation will be made orthogonal to the first $K-1$ approximate eigenfunctions. It may be shown that this procedure also yields an upper bound on the Kth eigenvalue.[11] The proof is based essentially on the principle that the class of functions which are used for the first $N+1$ approximate eigenfunctions is a narrower class of functions than the complete set of true eigenfunctions.

[11] Courant and Hilbert, *op. cit.*, ch. 6.

The substitution of a series of the functions ϕ_{En} into (19) and subsequent extremisation leads to a matrix eigenvalue problem. The resultant matrix is a symmetrical real matrix whose eigenvalues are approximations from above to the first $N+1$ eigenvalues. For each eigenvalue, a solution for an eigenvector exists and the totality of eigenvectors obtained form an orthogonal set with respect to suitable weighting factors. This latter result follows from the well-known theory of real symmetrical matrices.[12] For this reason, the orthogonalization conditions, which were originally imposed upon the functions ϕ_{En}, may be dispensed with.

From this point on, ψ_{En} and γ_n^2 will be used to denote the nth approximate eigenfunction and eigenvalue, respectively. For the Kth approximate eigenfunction take

$$\psi_{EK} = \sum_{n=0}^{N} a_{nK} \phi_{En} \quad (27)$$

where a_{nk} are unknown coefficients to be determined subject to the normalization condition

$$\int_0^b \kappa^{-1} \psi_{EK}^2 dy = \sum_{s=0}^{N} \sum_{n=0}^{N} a_{nK} a_{sK} P_{sn} = 1 \quad (28)$$

where

$$P_{sn} = P_{ns} = \int_0^b \kappa^{-1} \phi_{En} \phi_{Es} dy.$$

Substituting into (19) gives

$$\sum_{n=0}^{N} \sum_{s=0}^{N} \int_0^b \kappa^{-1} \left\{ \frac{d\phi_{Es}}{dy} \frac{d\phi_{En}}{dy} - \left(\kappa k_0^2 - \frac{\pi^2}{a^2} + \gamma_K^2 \right) \phi_{Es} \phi_{En} \right\} dy$$

$$= \text{stationary quantity.} \quad (29)$$

Let

$$\int_0^b \kappa^{-1} \left\{ \frac{d\phi_{Es}}{dy} \frac{d\phi_{En}}{dy} - \left(\kappa k_0^2 - \frac{\pi^2}{a^2} \right) \phi_{Es} \phi_{En} \right\} dy = T_{sn} = T_{ns}. \quad (30)$$

Thus,

$$\sum_{s=0}^{N} \sum_{n=0}^{N} a_{sK} a_{nK} (T_{sn} - \gamma_K^2 P_{sn}) = \text{stationary quantity.} \quad (31)$$

Equating all $\partial/\partial a_{nk}$ equal to zero for $n=0, 1, \cdots, N$, yields the following set of homogeneous equations

$$\sum_{n=0}^{N} a_{nK}(T_{sn} - \gamma_K^2 P_{sn}) = 0, \quad s = 0, 1, \cdots N. \quad (32)$$

[12] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, "Principles of Microwave Circuits," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 8, pp. 405–409; 1948.

For a solution, the determinant must vanish and this results in $N+1$ roots for γ_K^2 which are the $N+1$ approximate eigenvalues. For each root, say γ_K^2, a solution for a_{nK} can be obtained, this solution is unique when subjected to the normalization conditions (28). The set of coefficients a_{nK} are orthogonal to the set a_{nR} with respect to the weighting factors P_{sn} for $K \neq R$, *i.e.*,

$$\sum_{n=0}^{N} \sum_{s=0}^{N} a_{nR} a_{sK} P_{sn} = \delta_{RK}. \tag{33}$$

The proof is given in the Appendix.

For the LSE modes, the set of homogeneous equations obtained are

$$\sum_{n=1}^{N} b_{nK}(Q_{sn} - \beta_K^2 \delta_{sn}) = 0, \qquad s = 1, 2, \cdots N, \tag{34}$$

where

$$Q_{sn} = Q_{ns} = \int_0^b \left\{ \frac{d\phi_{Ms}}{dy} \frac{d\phi_{Mn}}{dy} - \left(\kappa k_0^2 - \frac{\pi^2}{a^2} \right) \phi_{Ms} \phi_{Mn} \right\} dy. \tag{35}$$

The vanishing of the determinant yields the first N approximate eigenvalues. The corresponding eigenvectors define the corresponding approximate eigenfunctions. The coefficients b_{nk} are subjected to the normalization condition

$$\sum_{n=1}^{N} b_{nK}^2 = 1, \qquad K = 1, 2, \cdots N,$$

and satisfy the orthogonality conditions

$$\sum_{n=1}^{N} b_{nK} b_{nR} = 0, \qquad R \neq K.$$

Matching of Fields at Junction

Consider the junction of an empty and inhomogeneously filled rectangular guide as illustrated in Fig. 2.

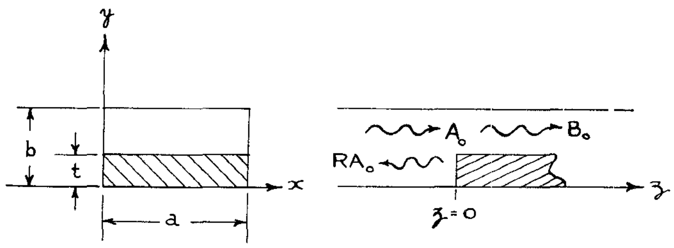

Fig. 2—Junction of an empty and partially filled rectangular guide.

Let an H_{10} mode be incident from the empty guide. The higher order modes excited will consist of an infinite number of the LSM and LSE modes. It will be assumed that only the dominant mode, *i.e.*, the first LSM mode, propagates in either the empty or partially filled guide. For an approximate solution, only a finite number of modes are considered and in the partially filled guide these will be taken as the approximate eigenfunctions as obtained by the R-R method. At the junction $z=0$, the tangential field components are made continuous and this leads to four simultaneous equations which must be solved for the reflection and transmission coefficients. The fields are obtained from the two Hertzian potentials $\vec{\Pi}_E$ and $\vec{\Pi}_M$ by means of (2), (7), (13), and (15). There are two different expressions for E_y and H_y which, however, yield the same results. In writing the continuity equations for the transverse field components, the functions of x and other common factors will be deleted to save space. The following equations, expressing continuity of E_y, H_x, E_x, and H_y, respectively, are obtained

$$\sqrt{\frac{1}{b}} \left(\Gamma_0^2 - \frac{\pi^2}{a^2} \right) (1 + R) A_0 + \sum_{n=1}^{N} A_n \left(\Gamma_n^2 - \frac{\pi^2}{a^2} \right) \sqrt{\frac{2}{b}} \cos \frac{n\pi}{b} y = \sum_{K=0}^{N} B_K \left(\gamma_K^2 - \frac{\pi^2}{a^2} \right) \sum_{n=0}^{N} a_{nK} \kappa^{-1} \sqrt{\frac{\epsilon_{0n}}{b}} \cos \frac{n\pi}{b} y, \tag{36}$$

$$j\omega\epsilon_0 \Gamma_0 (1 - R) \sqrt{\frac{1}{b}} A_0 - j\omega\epsilon_0 \sum_{n=1}^{N} A_n \Gamma_n \sqrt{\frac{2}{b}} \cos \frac{n\pi}{b} y - \sum_{s=1}^{N} \frac{s\pi^2}{ab} C_s \sqrt{\frac{2}{b}} \cos \frac{s\pi}{b} y = j\omega\epsilon_0 \sum_{K=0}^{N} B_K \gamma_K \sum_{n=0}^{N} a_{nK} \sqrt{\frac{2}{b}} \cos \frac{n\pi}{b} y - \sum_{K=1}^{N} \frac{\pi}{a} D_K \sum_{s=1}^{N} \frac{s\pi}{b} b_{sK} \sqrt{\frac{2}{b}} \cos \frac{s\pi}{b} y, \tag{37}$$

$$j\omega\mu \sum_{n=1}^{N} C_n \Gamma_n \sqrt{\frac{2}{b}} \sin \frac{n\pi}{b} y - \sum_{n=1}^{N} \frac{n\pi^2}{ab} A_n \sqrt{\frac{2}{b}} \sin \frac{n\pi}{b} y = -j\omega\mu \sum_{K=1}^{N} D_K \beta_K \sum_{n=1}^{N} b_{nK} \sqrt{\frac{2}{b}} \sin \frac{n\pi}{b} y - \sum_{K=0}^{N} B_K \sum_{n=1}^{N} \frac{n\pi^2}{ab} a_{nK} \kappa^{-1} \sqrt{\frac{2}{b}} \sin \frac{n\pi}{b} y, \tag{38}$$

$$\sum_{n=1}^{N} C_n \left(\Gamma_n^2 - \frac{\pi^2}{a^2} \right) \sqrt{\frac{2}{b}} \sin \frac{n\pi}{b} y = \sum_{K=1}^{N} D_K \left(\beta_K^2 - \frac{\pi^2}{a^2} \right) \sum_{n=1}^{N} b_{nK} \sqrt{\frac{2}{b}} \sin \frac{n\pi}{b} y, \tag{39}$$

where R is the complex reflection coefficient, A_n, B_n, C_n, and D_n are unknown amplitude coefficients, and

$$\Gamma_n^2 = \frac{\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} - k_0^2.$$

The amplitude of the dominant mode on the output side of the junction, *i.e.*, for $z>0$, is B_0.

Multiplying (36) and (37) by

$$\sqrt{\frac{\epsilon_{0n}}{b}} \cos \frac{n\pi}{b} y$$

and (38) and (39) by

$$\sqrt{\frac{2}{b}} \sin \frac{n\pi}{b} y$$

for $n=0, 1, \cdots, N$, in turn, and integrating from $y=0$ to $y=b$, converts these equations to the following algebraic equations

$$\left.\begin{aligned} -k_0^2(1+R)A_0 &= \sum_{K=0}^{N} B_K\left(\gamma_K^2 - \frac{\pi^2}{a^2}\right)\sum_{n=0}^{N} a_{nK}P_{n0}, \\ \left(\Gamma_n^2 - \frac{\pi^2}{a^2}\right)A_n &= \sum_{K=0}^{N} B_K\left(\gamma_K^2 - \frac{\pi^2}{a^2}\right)\sum_{s=0}^{N} a_{sK}P_{sn}, \quad n = 1, 2, \cdots N, \end{aligned}\right\} \tag{40}$$

$$\left.\begin{aligned} \Gamma_0(1-R)A_0 &= \sum_{K=0}^{N} B_K\gamma_K a_{0K}, \\ j\omega\epsilon_0\Gamma_n A_n + \frac{n\pi^2}{ab} C_n &= -j\omega\epsilon_0\sum_{K=0}^{N} B_K\gamma_K a_{nK} + \frac{n\pi^2}{ab}\sum_{K=1}^{N} D_K b_{nK}, \quad n = 1, 2, \cdots N, \end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned} j\omega\mu C_n\Gamma_n - \frac{n\pi^2}{ab}A_n &= -j\omega\mu\sum_{K=1}^{N} D_K\beta_K b_{nK} - \frac{n\pi^2}{ab}\sum_{K=0}^{N} B_K\sum_{s=1}^{N} a_{sK}J_{sn} \\ \text{where} \qquad J_{sn} &= \frac{b^2}{n^2\pi^2}\left[T_{sn} + k_0^2\delta_{sn} - \frac{\pi^2}{a^2}P_{sn}\right], \quad n = 1, 2, \cdots N, \end{aligned}\right\} \tag{42}$$

$$\left. C_n\left(\Gamma_n^2 - \frac{\pi^2}{a^2}\right) = \sum_{K=1}^{N} D_K\left(\beta_K^2 - \frac{\pi^2}{a^2}\right)b_{nK}, \quad n = 1, 2, \cdots N, \right\}. \tag{43}$$

This latter system of equations may be written in matrix form as a set of four homogeneous matrix equations. For a solution for the amplitude coefficients, the resultant determinant of the over-all system must vanish and this gives the value of the reflection coefficient directly. Alternatively, A_n and C_n may be eliminated by means of (40) and (43), leaving a system of two sets of equations involving B_k and D_k. The required solution for any particular case is obtained in a straightforward manner, but the general details are too lengthy for inclusion here.

An Example

The reflection and transmission coefficients were calculated for an H_{10} mode incident on a partially filled guide, as illustrated in Fig. 2, for values of t/b ranging from 0 to 1. The free space wavelength was 3.14 cm, the dielectric constant was 2.52, and the internal waveguide dimensions were 0.9×0.4 inch. The dominant mode and two evanescent modes (one LSE and one LSM mode) were taken into account in both the empty and partially filled guide. This led to a sixth-order determinant which, however, had fifteen of its elements equal to zero, and was readily reduced to a third-order determinant. This latter determinant gave $(1-R)/(1+R)$ as the ratio of two second-order determinants. In Fig. 3 opposite, the modulus and phase angle of the reflection coefficient are plotted, while the phase angle of the transmitted wave is plotted in Fig. 4. The computed values of the transmission coefficient phase angle are not very accurate, because of their small absolute value. The measured values are also plotted in the above figures and in all cases are within the estimated experimental error from the computed values. The measured values were obtained by the usual tangent method, *i.e.*, by plotting the field minimum position in the partially filled guide vs short circuit position in the empty guide and subsequent analysis of the resultant curve. The dielectric slab was located in the slotted standing-wave detector section.

It is interesting to note that the modulus of the reflection coefficient is within one or two per cent of what

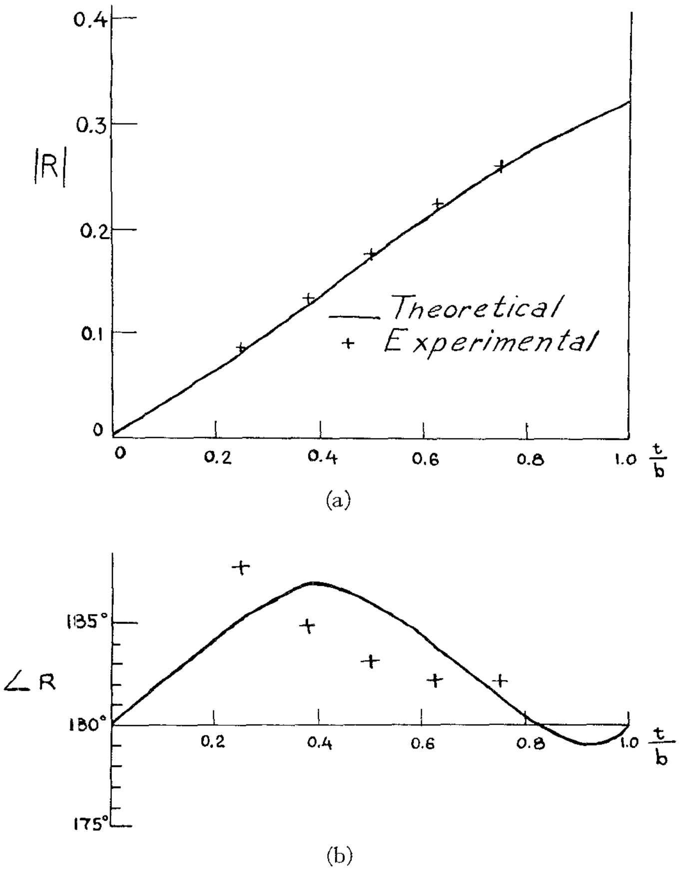

Fig. 3—(a) Modulus of reflection coefficient, (b) phase angle of reflection coefficient.

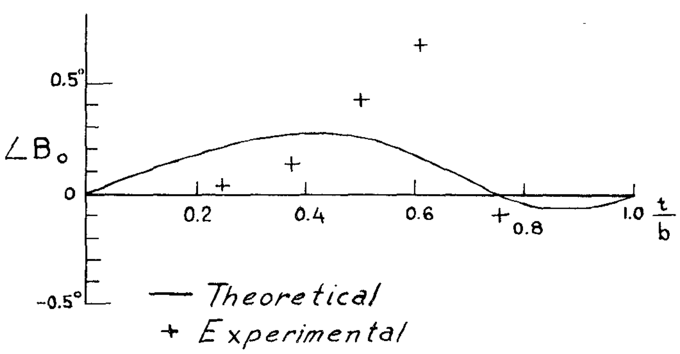

Fig. 4—Phase angle of transmitted wave.

one would compute by assuming that the junction is equivalent to a junction of two transmission lines with characteristic impedances proportional to the respective wavelengths in the empty and partially filled guides. From Fig. 3(b), it is seen that for values of $t/b<0.82$, the phase angle of the reflection coefficient is greater than π radians, while for values of $t/b>0.82$ the phase angle is less than π radians, corresponding respectively to more electric energy than magnetic energy stored in the evanescent modes at the junction and vice versa. The evanescent LSE modes store more magnetic energy, while the evanescent LSM modes store more electric energy, except when the modes are close to propagating, so that

$$\gamma_n^2 < \frac{\pi^2}{a^2} \quad \text{and} \quad \beta_n^2 < \frac{\pi^2}{a^2}\cdot$$

For values of

$$\frac{t}{b} > 0.85, \qquad \gamma_1^2 < \frac{\pi^2}{a^2}$$

and the first evanescent LSM mode stores an excess of magnetic energy at the junction, resulting in a phase angle of less than π radians for the reflection coefficient.

These results are obtained by considering only a small number of modes. If a larger number of modes are taken into account, sufficient compensation may take place so that the phase angle of the reflection coefficient does not become less than π radians for this particular sample. Further calculations are required to clarify this behavior.

The modulus of the transmission coefficient can be computed only when the characteristic impedances of the empty and partially filled guide have been specified. Since these may be specified in any convenient way, the modulus of the transmission coefficient is not unique, the only restriction being that the transmitted power must be equal to the difference between the incident and reflected power.

Conclusion

The use of the R-R method for obtaining approximations to the first N eigenfunctions in a partially filled guide permits one to evaluate, in a straightforward manner, the junction discontinuity existing between an empty and partially filled guide. The reduction in computational labor is considerable, and the solution of transcendental equations and the evaluation of many complex expressions is avoided. The method outlined here may equally well be applied to the evaluation of the parameters of a slotted dielectric interface in free space.

Appendix

Eq. (32) in the text may be written in matrix form as follows

$$\begin{bmatrix} T_{00} & T_{01} & \cdots & T_{0N} \\ \cdot & & & \\ \cdot & & & \\ T_{N0} & \cdot & \cdots & T_{NN} \end{bmatrix} \begin{bmatrix} a_{0K} \\ \cdot \\ \cdot \\ a_{NK} \end{bmatrix} = \gamma_K^2 \begin{bmatrix} P_{00} & \cdots & P_{0N} \\ \cdot & & \\ P_{N0} & \cdots & P_{NN} \end{bmatrix} \begin{bmatrix} a_{0K} \\ \cdot \\ a_{NK} \end{bmatrix} \qquad (44)$$

or more briefly as

$$[T_{ij}][a_{jK}] = \gamma_K^2[P_{ij}][a_{jK}] \tag{45}$$

and for the Rth solution as

$$[T_{ij}][a_{jR}] = \gamma_R^2[P_{ij}][a_{jR}]. \tag{46}$$

Take the transpose of (2) to get

$$\{a_{jK}\}[T_{ji}] = \gamma_K^2\{a_{jK}\}[P_{ji}],$$

or

$$\{a_{jK}\}[T_{ij}] = \gamma_K^2\{a_{jK}\}[P_{ij}], \tag{47}$$

since $T_{ij}=T_{ji}$ and $P_{ij}=P_{ji}$ and where $\{a_{jk}\}$ is a row matrix. Postmultiply (4) by $[a_{jR}]$, premultiply (3) by $\{a_{jk}\}$, and subtract to get

$$(\gamma_K^2 - \gamma_R^2)\{a_{jK}\}[P_{ij}][a_{jR}] = 0. \tag{48}$$

When $\gamma_K^2 \neq \gamma_R^2$, (5) gives

$$\sum_{n=0}^{N}\sum_{s=0}^{N} a_{nK}a_{sR}P_{sn} = 0,$$

upon development of the matrix product. This proves the orthogonality of the eigenvectors.

Coupling Through an Aperture Containing an Anisotropic Ferrite*

DONALD C. STINSON†

Summary—**Coupling through an aperture containing anisotropic ferrites is investigated theoretically by a simple extension of Bethe's small-hole coupling theory to include the dipole moment of the body in the aperture. The magnetic dipole moment of the ferrite body is ordinarily a vector but becomes a tensor upon the application of a magnetostatic field. This new theory is applicable to any situation where Bethe's small-hole coupling theory is valid. Experimental verification was quite satisfactory and was obtained on two Bethe-hole type couplers: one with the waveguides parallel, and the other with the waveguides perpendicular.**

Introduction

THE THEORY of coupling through small windows was formulated by Bethe more than a decade ago.[1] Initially, he found that the amplitudes of the modes excited in a waveguide by a window were proportional to

$$\int \overline{E}_1 \times \overline{H}_2 \cdot \bar{n} ds$$

where field 1 is the excited field, field 2 is a normal mode of the guide, and $\bar{n}$ is the inward normal. Later, he evaluated the integral over the window by developing a lumped-constant theory[2] for small windows and then applied this lumped-constant theory to side windows[3] in waveguides.

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† Lockheed Aircraft Corp., Sunnyvale, Calif.

[1] H. A. Bethe, "Formal Theory of Waveguides of Arbitrary Cross Section," M.I.T. Rad. Lab. Rep. 43-26; March 16, 1943.

[2] H. A. Bethe, "Lumped Constants for Small Irises," M.I.T. Rad. Lab. Rep. 43-22; March 24, 1943.

[3] H. A. Bethe, "Theory of Side Windows in Wave Guides," M.I.T. Rad. Lab. Rep. 43-27; April 4, 1943.

Bethe's coupling theory depends upon his lumped-constant theory for small windows, which in turn depends upon replacing the excitation caused by the window by a quantity which is proportional to the following parameters: 1) frequency; 2) the normal electric or tangential magnetic field (exciting field) which would exist at the center of gravity of the window if the window were replaced by a solid metal wall; 3) the corresponding fields (induced fields) of the normal modes which are excited by the window; and 4) lumped constants (polarizabilities) which are functions only of the shape and dimensions of the window. The basis of his lumped-constant theory depends upon the fact that the excitation of the window can be replaced by "equivalent" electric and magnetic dipole moments. These "equivalent" electric and magnetic dipole moments lead him to consider the polarizabilities (which are defined as the "equivalent" dipole moments per unit incident field) as the true lumped constants of the window. This is logical since a window may act as either an inductive or capacitive element, depending upon its location and the propagating mode in the waveguide.

Since his coupling theory applies only to cases where the window and the waveguides are filled with the same isotropic and homogeneous material, it is the purpose of this paper to extend his theory to include cases where the window is completely filled with an anisotropic ferrite. The ferrite involved is anisotropic in the sense that its permeability becomes a tensor upon the application of a magnetostatic field. This extension will be made by adding the "equivalent" magnetic dipole moment of the ferrite to that of the window.